Int. J. Solids Structures, 1969, Vol. 5, pp. 915 to 920. Pergamon Press. Printed in Great Britain

THE INFLUENCE OF RANDOM POISSON'S RATIO ON DISPLACEMENTS IN AN ELASTIC HALF-PLANE*

H. BARGMANN and J. GOLECKI†

II. Institut für Mechanik
Technische Hochschule,
Vienna

Abstract—Displacement boundary problems are considered for the homogeneous isotropic elastic half-plane with random Poisson's ratio ν. Probability densities, expected values and variances of the displacement field u, v are determined and evaluated for uniform probability distribution of ν. As an example the effect of a discontinuity of the boundary vertical displacement $v_0 = v_0(x, 0)$ is determined and used to obtain the settlement of the earth surface under a coal excavation.

1. INTRODUCTION

THE mechanical behavior of a homogeneous, isotropic and linear-elastic solid is described by the moduli of elasticity such as Young's modulus E, Kirchhoff's modulus G, bulk modulus K and Poisson's ratio ν. The classical methods to determine these moduli are based on tensile or compression tests with simultaneous measurement of longitudinal and transverse deformations. These tests provide direct determination of Young's modulus and Poisson's ratio. From test results E, G, K and ν are calculated usually as deterministic quantities by using some kind of arithmetical mean value. In many instances this may provide an adequate description. In other cases, however, this will not give sufficient information. Since the results are, in fact, varying randomly the elastic moduli should be treated as random quantities.

Frequently, e.g. from rock tests, Young's modulus is obtained as a constant (or its small variation may be neglected), whereas Poisson's ratio is a random variable varying in the interval $[0, \frac{1}{2}]$. In this paper the influence of the randomness of ν on the displacements of an elastic half-plane under displacement boundary conditions is considered. The problem is formulated to explain certain phenomena in geology and rock mechanics [1, 3, 5].

2. DISPLACEMENTS IN THE ELASTIC HALF-PLANE

The displacement field u, v in the elastic half-plane under displacement boundary conditions can be described in the form [1, 2]

$$
\begin{aligned}
u &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left(1-\frac{1}{3-4\nu'}y|\xi|\right)U+\frac{1}{3-4\nu'}iy\xi V\right\}e^{-|\xi|y-i\xi x}\,d\xi,\\
v &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{3-4\nu'}iy\xi U+\left(1+\frac{1}{3-4\nu'}y|\xi|\right)V\right\}e^{-|\xi|y-i\xi x}\,d\xi,
\end{aligned}
\tag{2.1}
$$

* The research reported in this paper has been sponsored in part by the United States Government.
† Now at: Faculty of Civil Engineering, Technion–Israel Institute of Technology, Haifa.

where ν' stands for the generalized Poisson's ratio ($\nu' = \nu$ or $\nu' = \nu/(1+\nu)$ in plane strain or plane stress, respectively) and U, V denote the Fourier transforms of the boundary displacements $u_0 \equiv u(x,0), v_0 \equiv v(x,0)$,

$$U = \int_{-\infty}^{\infty} u_0 \, e^{i\xi x} \, dx, \qquad V = \int_{-\infty}^{\infty} v_0 \, e^{i\xi x} \, dx. \tag{2.2}$$

Letting $u_i = (u, v)$, $U_i = (U, V)$, the displacements may be written in the form,
plain strain

$$u_i = A_{u_i} + B_{u_i} \frac{1}{3-4\nu} \equiv g_{u_i}(\nu), \tag{2.3}$$

plane stress

$$u_i = A_{u_i} + B_{u_i} \frac{1+\nu}{3-\nu} \equiv g_{u_i}(\nu), \tag{2.4}$$

where

$$\begin{aligned} A_{u_i} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U_i \, e^{-|\xi| y - i\xi x} \, d\xi, \\ B_u &= \frac{y}{2\pi} \int_{-\infty}^{\infty} (i\xi V - |\xi| U) \, e^{-|\xi| y - i\xi x} \, d\xi, \\ B_v &= \frac{y}{2\pi} \int_{-\infty}^{\infty} (i\xi U + |\xi| V) \, e^{-|\xi| y - i\xi x} \, d\xi, \end{aligned} \tag{2.5}$$

are deterministic quantities.

Let the probability density $p(\nu)$ of Poisson's ratio ν be given. Then the corresponding densities of the random displacements are obtained as [4]

$$q(u_i) = \frac{p(\nu_1)}{|g'(\nu_1)|}, \tag{2.6}$$

where $g(\nu) \equiv g_{u_i}(\nu)$ represents the functions in equations (2.3), (2.4) $g'(\nu) = dg(\nu)/d\nu$, and ν_1, depending on u_i, is the (real) root of $u_i = g(\nu)$.

Thus, one obtains for the densities,
plane strain

$$q(u_i) = \frac{|B|}{4(u_i - A)^2} \cdot p\left(\frac{3}{4} - \frac{B}{4(u_i - A)}\right), \tag{2.7}$$

plane stress

$$q(u_i) = \frac{4|B|}{(u_i - A + B)^2} \cdot p\left(\frac{3u_i - 3A - B}{u_i - A + B}\right), \tag{2.8}$$

when A, B are to be read as A_{u_i}, B_{u_i}, respectively.

The expected value (mean) m of the displacements and the variance σ^2,

$$\begin{aligned} m &\equiv \langle u_i \rangle = \int_{-\infty}^{\infty} u_i q(u_i) \, du_i, \\ \sigma^2 &\equiv \langle (u_i - m)^2 \rangle = \int_{-\infty}^{\infty} (u_i - m)^2 q(u_i) \, du_i, \end{aligned} \tag{2.9}$$

may be found directly in terms of the density $p(\nu)$ of ν, since

$$m = \langle g(\nu)\rangle = \int_{-\infty}^{\infty} g(\nu)p(\nu)\,d\nu,$$
$$\sigma^2 = \langle u_i^2\rangle - m^2. \tag{2.10}$$

Assuming now that ν is uniformly distributed in the interval $[a, b]$, $a < b$, $0 \le a, b \le \frac{1}{2}$,

$$p(\nu) = \begin{cases} 1/(b-a) & \text{for } a \le \nu \le b \\ \text{zero elsewhere,} & \end{cases} \tag{2.11}$$

one obtains the densities,
plane strain

$$q(u_i) = \frac{|B|}{4(b-a)(u_i-A)^2} \quad \text{for } A + \frac{B}{3-4a} \lesseqgtr u_i \lesseqgtr A + \frac{B}{3-4b} \tag{2.12}$$

and zero elsewhere,
plane stress

$$q(u_i) = \frac{4|B|}{(b-a)(u-A+B)^2} \quad \text{for } A + B\frac{1+a}{3-a} \lesseqgtr u_i \lesseqgtr A + B\frac{1+b}{3-b} \tag{2.13}$$

and zero elsewhere. The upper and lower signs correspond to $B > 0$, $B < 0$, respectively.

Expected values and variances are obtained as,
plane strain

$$\langle u_i\rangle = A - \frac{B}{4(b-a)}\ln\left|\frac{3-4b}{3-4a}\right|,$$
$$\sigma_{u_i}^2 = B^2\left\{\frac{1}{(3-4a)(3-4b)} - \frac{1}{16(b-a)^2}\ln^2\left|\frac{3-4b}{3-4a}\right|\right\} \tag{2.14}$$

plane stress

$$\langle u_i\rangle = A - B\left\{1 + \frac{4}{(b-a)}\ln\left|\frac{3-b}{3-a}\right|\right\},$$
$$\sigma_{u_i}^2 = 16B^2\left\{\frac{1}{(3-a)(3-b)} - \frac{1}{(b-a)^2}\ln^2\left|\frac{3-b}{3-a}\right|\right\}. \tag{2.15}$$

The densities as derived above provide sufficient information in many applications. For instance, the probability that the displacement u_i will lie between two limits ζ_1, ζ_2, is

$$P\{\zeta_1 \le u_i \le \zeta_2\} = \int_{\zeta_1}^{\zeta_2} q(u_i)\,du_i. \tag{2.16}$$

The inverse problem, where the probability P is prescribed and the corresponding bounds ζ_1, ζ_2 are to be determined can, in general, only be solved numerically.

In a similar manner as above the statistical properties of the stresses can be calculated.

3. DISPLACEMENT FIELD IN THE NEIGHBORHOOD OF A VERTICAL FAULT

Let the boundary displacements of the half-plane ($y \geq 0$) be

$$u_0 = 0 \quad \text{for } -\infty < x < \infty,$$
$$v_0 = \begin{cases} -h_0 & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases} \tag{3.1}$$

in $y = 0$.

This represents a boundary in the form of a vertical fault. The Fourier transforms of the boundary displacements become [3]

$$U = 0, \qquad V = -h_0\left[\frac{i}{\xi}+\pi\delta(\xi)\right], \tag{3.2}$$

where $\delta(\xi)$ is Dirac's delta function.

The displacements u, v in a neighborhood of the fault are (ν is independent of x, y) from equations (2.1), [3],

$$u = \frac{h_0}{\pi(3-4\nu')}\frac{y^2}{x^2+y^2},$$
$$v = -h_0\left\{\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{y}+\frac{1}{\pi(3-4\nu')}\frac{xy}{x^2+y^2}\right\}, \tag{3.3}$$

$y > 0$.

Now let ν be a random variable (independent of x, y) distributed according to equation (2.11). After calculating the deterministic parts, equations (2.5), $y > 0$,

$$A_u = 0, \qquad B_u = \frac{h_0}{\pi}\frac{y^2}{x^2+y^2},$$
$$A_v = -h_0\left(\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{y}\right), \qquad B_v = -\frac{h_0}{\pi}\frac{xy}{x^2+y^2}, \tag{3.4}$$

one obtains for the expected values and variances, equations (2.14), (2.15), respectively, plane strain

$$\langle u\rangle = -\frac{h_0}{4\pi(b-a)}\ln\left|\frac{3-4b}{3-4a}\right|\frac{y^2}{x^2+y^2},$$
$$\langle v\rangle = -h_0\left\{\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{y}-\frac{1}{4\pi(b-a)}\ln\left|\frac{3-4b}{3-4a}\right|\frac{xy}{x^2+y^2}\right\},$$
$$\sigma_u^2 = \frac{h_0^2}{\pi^2}\left\{\frac{1}{(3-4a)(3-4b)}-\frac{1}{16(b-a)^2}\ln^2\left|\frac{3-4b}{3-4a}\right|\right\}\frac{y^4}{(x^2+y^2)^2},$$
$$\sigma_v^2 = \frac{h_0^2}{\pi^2}\left\{\frac{1}{(3-4a)(3-4b)}-\frac{1}{16(b-a)^2}\ln^2\left|\frac{3-4b}{3-4a}\right|\right\}\frac{x^2y^2}{(x^2+y^2)^2}, \tag{3.5}$$

plane stress

$$
\begin{aligned}
\langle u\rangle &= -\frac{h_0}{\pi}\left\{1+\frac{4}{(b-a)}\ln\left|\frac{3-b}{3-a}\right|\right\}\frac{y^2}{x^2+y^2},\\
\langle v\rangle &= -h_0\left\{\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{y}-\frac{1}{\pi}\left[1+\frac{4}{(b-a)}\ln\left|\frac{3-b}{3-a}\right|\right]\cdot\frac{xy}{x^2+y^2}\right\},\\
\sigma_u^2 &= 16\frac{h_0^2}{\pi^2}\left\{\frac{1}{(3-a)(3-b)}-\frac{1}{(b-a)^2}\ln^2\left|\frac{3-b}{3-a}\right|\right\}\frac{y^4}{(x^2+y^2)^2},\\
\sigma_v^2 &= 16\frac{h_0^2}{\pi^2}\left\{\frac{1}{(3-a)(3-b)}-\frac{1}{(b-a)^2}\ln^2\left|\frac{3-b}{3-a}\right|\right\}\frac{x^2y^2}{(x^2+y^2)^2}.
\end{aligned}
\tag{3.6}
$$

Consider as an example the deformation of the earth surface due to a coal excavation, when Poisson's ratio ν is uniformly distributed in the interval $[0, \frac{1}{2}]$. Assume the excavation of large width L_w and length L to be situated at the large depth H. As a model for the calculation of the displacements one may in this case use a half-plane under boundary displacements equation (3.1) and assume plane strain, $\nu' = \nu$, [5].

Then, with $a = 0$, $b = \frac{1}{2}$, equations (3.5), yield

$$
\begin{aligned}
\langle u\rangle &= \ln 3\frac{h_0}{2\pi}\frac{y^2}{x^2+y^2},\\
\langle v\rangle &= -h_0\left\{\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{y}+\frac{\ln 3}{2\pi}\frac{xy}{x^2+y^2}\right\},\\
\sigma_u^2 &= \frac{h_0^2}{12\pi^2}(4-3\ln^2 3)\frac{y^4}{(x^2+y^2)^2},\\
\sigma_v^2 &= \frac{h_0^2}{12\pi^2}(4-3\ln^2 3)\frac{x^2y^2}{(x^2+y^2)^2}.
\end{aligned}
\tag{3.7}
$$

For comparison the displacements in this case are also calculated in the usual way, with Poisson's ratio taken deterministically as arithmetical mean value, $\nu = 0{\cdot}25$, equations (3.3),

$$
\begin{aligned}
u_d &= \frac{h_0}{2\pi}\frac{y^2}{x^2+y^2},\\
v_d &= -h_0\left\{\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{y}+\frac{1}{2\pi}\frac{xy}{x^2+y^2}\right\}.
\end{aligned}
\tag{3.8}
$$

Thus, the difference between expected value and deterministic mean is for the horizontal displacement

$$
\langle u\rangle - u_d = (\ln 3-1)\frac{h_0}{2\pi}\frac{y^2}{x^2+y^2} \doteq 0{\cdot}09861\frac{h_0}{2\pi}\frac{y^2}{x^2+y^2},
\tag{3.9}
$$

corresponding to a relative error of about 10%, i.e. the displacement u, which has to be expected, is about 10% larger than its "deterministic" value. This is of importance when there is an upper bound restriction on the horizontal displacements.

For the vertical displacements one obtains

$$\langle v\rangle - v_d = -(\ln 3-1)\frac{h_0}{2\pi}\frac{xy}{x^2+y^2} \doteq -0{\cdot}09861\frac{h_0}{2\pi}\frac{xy}{x^2+y^2}. \tag{3.10}$$

For $x = y$ there is a maximum difference of about 0·01 h_0. Here, in many cases the deterministic calculation will give sufficient results. The last equation shows, for $y = H = \text{const}$, the approximate error in the determination of the settlement surface.

REFERENCES

[1] J. GOLECKI, Approximate method of determining the distribution of stress in the neighbourhood of folds. *Bull. Acad. pol. Sci. Sér. Sci. tech.* **9**, 383 (1961).
[2] J. GOLECKI, Elastic half-plane with variable Poisson's ratio. Displacement boundary problems. *Bull. Acad. pol. Sci. Sér. Sci. tech.* **16**, 175 (1968).
[3] J. GOLECKI and S. JÓŻKIEWICZ, Distribution of displacements and stresses in the neighbourhood of a vertical fault. *Bull. Acad. pol. Sci. Sér. Sci. tech.* **9**, 447 (1961).
[4] A. PAPOULIS, *Probability, Random Variables and Stochastic Processes*. McGraw-Hill (1965).
[5] J. GOLECKI and S. JÓŻKIEWICZ, Distribution of displacements and stresses in the neighbourhood of two vertical faults. *Archwm Gorn.* **7**, 27 (1962) (in Polish).

(*Received* 16 *January* 1969)

Абстракт—Рассматриваются граничные задачи в перемещениях для однородной изотропной упругой полуплоскости, с произвольно выбранным коэффициентом Пуассона ν. Определяются и оцениваются плотности вероятности, ожидаемые значения и изменения поля перемещений u, v для однородного распределения вероятности ν. В качестве примера определяется эффект разрыва на границе вертикального перемещения $v_0 = v_0(x, u)$. Далее используется его для получения оседания дневной поверхности под влиянием выработки угля.